

# Global existence and asymptotic behaviour in the future for the Einstein-Vlasov system with positive cosmological constant

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## Abstract

The behaviour of expanding cosmological models with collisionless matter and a positive cosmological constant is analysed. It is shown that under the assumption of plane or hyperbolic symmetry the area radius goes to infinity, the spacetimes are future geodesically complete, and the expansion becomes isotropic and exponential at late times. This proves a form of the cosmic no hair theorem in this class of spacetimes.

## 1 Introduction

The presence of a positive cosmological constant  $\Lambda$  can lead to exponential expansion in cosmological models. This is the simplest mathematical description of an inflationary universe. Under certain circumstances the de Sitter solution acts as a late time attractor for more general solutions of the Einstein equations with  $\Lambda > 0$ . This is sometimes known as the cosmic no hair theorem. Up to now there are unfortunately not many cases where this kind of statement has been proved rigorously for inhomogeneous spacetimes.

A positive cosmological constant can be introduced in Newtonian cosmology and this provides a simplified model for the general relativistic case. In [4] a form of the cosmic no hair theorem was proved in Newtonian cosmology. A perfect fluid was used as a matter model and solutions were considered which evolve from initial data which are small but finite perturbations of homogeneous data. It was shown that if the homogeneous solution exists globally in the future the same is true of the inhomogeneous solution. Of course a global in time existence theorem is a prerequisite for a proof

of the cosmic no hair theorem. It was then shown that the inhomogeneous solutions have a behaviour at late times which is qualitatively similar to that of the homogeneous model. If  $\bar{\rho}$  is the mean density and  $\delta\rho = \rho - \bar{\rho}$  then  $\delta\rho/\bar{\rho}$  converges as  $t \rightarrow \infty$ . In Newtonian cosmology there is also a theorem about the late time asymptotics of models with a kinetic description of matter by the Vlasov equation and with vanishing cosmological constant [8]. The boundedness of  $\delta\rho/\bar{\rho}$  is also obtained in that case. Adding a positive cosmological constant to the problem considered in [8] would presumably simplify the analysis but this has not been attempted.

In general relativity the problem of proving the cosmic no hair theorem is more difficult. In the spatially homogeneous case there is a general result of Wald [12] on spacetimes with positive cosmological constant which does not depend on the details of the matter content but only on energy conditions. There is one example where future geodesic completeness has been proved for a class of inhomogeneous spacetimes with matter [7]. This concerns solutions of the Einstein-Vlasov system with hyperbolic symmetry and  $\Lambda = 0$  satisfying an additional inequality on the initial data. Under those assumptions geodesic completeness was proved but only limited information was obtained on the asymptotic behaviour at late times. In the following we will show that in the presence of a positive cosmological constant this result can be strengthened a lot. Future geodesic completeness is proved for all solutions with hyperbolic and plane symmetry. Moreover the asymptotic behaviour is shown to closely resemble that of the de Sitter solution. It should be mentioned that in the case of the vacuum Einstein equations there is a proof of a form of the cosmic no hair theorem which does not require symmetry assumptions but does require a small data restriction [5].

In this paper we study solutions of the Einstein equations with positive cosmological constant coupled to the Vlasov equation describing collisionless matter. Under the assumption of plane or hyperbolic symmetry we show that the solutions are future geodesically complete and we obtain a detailed description of their late time behaviour, which is similar to that of the de Sitter solution. The proof is built on the local existence theorem and continuation criterion previously proved by Tchapnda and Noutchegueme [11].

Let us recall the formulation of the Einstein-Vlasov system which governs the time evolution of a self-gravitating collisionless gas in the context of general relativity; for the moment we do not assume any symmetry of the spacetime. All the particles in the gas are assumed to have equal rest mass, normalized to unity. The four-momentum of each particle is a future-pointing unit timelike vector so that the number density  $f$  of particles is a non-negative function supported on the mass shell

$$PM := \{g_{\alpha\beta}p^\alpha p^\beta = -1, p^0 > 0\},$$

a submanifold of tangent bundle  $TM$  of the space-time manifold  $M$  with

metric  $g$  (the signature is  $- + ++$ ). We use coordinates  $(t, x^a)$  with zero shift and corresponding canonical momenta  $p^\alpha$ ; Greek indices always run from 0 to 3, and Latin ones from 1 to 3. On the mass shell  $PM$  the variable  $p^0$  becomes a function of the remaining variables  $(t, x^a, p^b)$ :

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + g_{ab} p^a p^b}.$$

The Einstein-Vlasov system now reads

$$\begin{aligned} \partial_t f + \frac{p^a}{p^0} \partial_{x^a} f - \frac{1}{p^0} \Gamma_{\beta\gamma}^a p^\beta p^\gamma \partial_{p^a} f &= 0 \\ G_{\alpha\beta} + \Lambda g_{\alpha\beta} &= 8\pi T_{\alpha\beta} \\ T_{\alpha\beta} &= - \int_{\mathbb{R}^3} f p_\alpha p_\beta |g|^{1/2} \frac{dp^1 dp^2 dp^3}{p_0} \end{aligned}$$

where  $p_\alpha = g_{\alpha\beta} p^\beta$ ,  $\Gamma_{\beta\gamma}^\alpha$  are the Christoffel symbols,  $|g|$  denotes the determinant of the metric  $g$ ,  $G_{\alpha\beta}$  the Einstein tensor,  $\Lambda$  the cosmological constant, and  $T_{\alpha\beta}$  is the energy-momentum tensor.

In this paper we want to investigate the Einstein-Vlasov system with positive cosmological constant and plane or hyperbolic symmetry in the expanding direction. For the notion of spherical, plane or hyperbolic symmetry, we refer to [9]. We write the system in areal coordinates, i.e., the coordinates are chosen such that  $R = t$ , where  $R$  is the area radius function on a surface of symmetry.

The circumstances under which coordinates of this type exist are discussed in [2] for the Einstein-Vlasov system with vanishing cosmological constant. It will now be shown that the analysis there can be extended to the situation under consideration here. Consider first the Einstein equations with a general matter model satisfying the dominant energy condition and  $\Lambda = 0$ . For plane symmetric spacetimes it follows from Proposition 3.1 of [10] that the gradient of  $R$  is always timelike. The corresponding statement in the case of hyperbolic symmetry can be proved by the argument in Step 1 in section 4 of [2]. When there is a cosmological constant we can consider it as a fictitious matter field with ‘energy-momentum tensor’  $-\Lambda g_{\alpha\beta}$ . This fictitious matter field satisfies the dominant energy condition. The same is true of the tensor which is the sum of the fictitious energy-momentum tensor with the energy-momentum tensor of real matter satisfying the dominant energy condition. It can be concluded from all this that the gradient of  $R$  is timelike for the Einstein-Vlasov system with positive cosmological constant and plane or hyperbolic symmetry. The remainder of Step 1 and Step 2 in section 4 of [1] and [2] can be extended to the case where a positive cosmological constant is present using the same method of the fictitious energy-momentum tensor. The only property which is required in addition to the dominant energy condition is the inequality  $q \leq \rho - p$  and this is satisfied by the fictitious energy-momentum tensor. From this point it is possible

to argue exactly as in [1] and [2] to conclude that a solution of the Einstein-Vlasov system with positive cosmological constant and plane or hyperbolic symmetry contains a Cauchy surface where  $R$  is constant. Hence there is no loss of generality in restricting consideration to spacetimes evolving from a hypersurface of constant areal time.

Since the gradient of  $R$  is everywhere timelike it must be either everywhere future-pointing timelike or everywhere past-pointing timelike. We choose a time orientation such that the latter is the case. Then the expanding direction of the cosmological model corresponding to increasing area radius  $t$ .

The metric takes the form

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + t^2(d\theta^2 + \sin_k^2\theta d\varphi^2) \quad (1.1)$$

Here  $t > 0$ , the functions  $\lambda$  and  $\mu$  are periodic in  $r$  with period 1 and

$$\sin_k \theta := \begin{cases} \sin \theta & \text{if } k = 1 \\ 1 & \text{if } k = 0 \\ \sinh \theta & \text{if } k = -1 \end{cases}$$

For the case  $k = 1$  the orbits of the symmetry action are two-dimensional spheres. In this spherically symmetric case, as well as in the case  $\Lambda < 0$ , the global result below is seen to be false, cf. [11], so these cases will not be considered further. For the plane symmetric case  $k = 0$  the orbits of the symmetry action are flat tori, they are hyperbolic spaces for the hyperbolic symmetry  $k = -1$ . The coordinates  $(\theta, \varphi)$  range in  $[0, 2\pi] \times [0, 2\pi]$  or  $[0, \infty[ \times [0, 2\pi]$  for  $k = 0, -1$  respectively. It has been shown in [6] and [2] that due to the symmetry  $f$  can be written as a function of

$$t, r, w := e^\lambda p^1 \text{ and } F := t^4(p^2)^2 + t^4 \sin_k^2 \theta (p^3)^2,$$

i.e.  $f = f(t, r, w, F)$ . In these variables we have  $p^0 = e^{-\mu} \sqrt{1 + w^2 + F/t^2}$ . After calculating the Vlasov equation in these variables, the non-trivial components of the Einstein tensor, and the energy-momentum tensor and denoting by a dot or by prime the derivatives of the metric components with respect to  $t$  or  $r$  respectively, the complete Einstein-Vlasov system reads as follows :

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{1 + w^2 + F/t^2}} \partial_r f - (\dot{\lambda} w + e^{\mu-\lambda} \mu' \sqrt{1 + w^2 + F/t^2}) \partial_w f = 0 \quad (1.2)$$

$$e^{-2\mu} (2t\dot{\lambda} + 1) + k - \Lambda t^2 = 8\pi t^2 \rho \quad (1.3)$$

$$e^{-2\mu} (2t\dot{\mu} - 1) - k + \Lambda t^2 = 8\pi t^2 p \quad (1.4)$$

$$\mu' = -4\pi t e^{\lambda+\mu} j \quad (1.5)$$

$$e^{-2\lambda} (\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu} \left( \ddot{\lambda} + (\dot{\lambda} - \dot{\mu})(\dot{\lambda} + \frac{1}{t}) \right) + \Lambda = 4\pi q \quad (1.6)$$

where

$$\rho(t, r) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} f(t, r, w, F) dF dw = e^{-2\mu} T_{00}(t, r) \quad (1.7)$$

$$p(t, r) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{1+w^2+F/t^2}} f(t, r, w, F) dF dw = e^{-2\lambda} T_{11}(t, r) \quad (1.8)$$

$$j(t, r) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t, r, w, F) dF dw = -e^{\lambda+\mu} T_{01}(t, r) \quad (1.9)$$

$$q(t, r) := \frac{\pi}{t^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F}{\sqrt{1+w^2+F/t^2}} f(t, r, w, F) dF dw = \frac{2}{t^2} T_{22}(t, r). \quad (1.10)$$

We prescribe initial data at some time  $t = t_0 > 0$ ,

$$f(t_0, r, w, F) = \overset{\circ}{f}(r, w, F), \quad \lambda(t_0, r) = \overset{\circ}{\lambda}(r), \quad \mu(t_0, r) = \overset{\circ}{\mu}(r).$$

The organization of the paper is as follows. In the next section we show that the solution of the initial value problem corresponding to the Einstein-Vlasov system (1.2)-(1.6) exists for all  $t \geq t_0$ . In section 3 we first prove that the spacetime obtained in section 2 is timelike and null geodesically complete towards the future; later on we determine the explicit leading behaviour of solution, compute the generalized Kasner exponents and prove that each of them tends to  $\frac{1}{3}$  as  $t$  tends to  $+\infty$ .

## 2 Global existence in the future

In this section we follow the approach of [2]. We make use of the continuation criterion in the following local existence result :

**Proposition 2.1** *Let  $\overset{\circ}{f} \in C^1(\mathbb{R}^2 \times [0, \infty[)$  with  $\overset{\circ}{f}(r+1, w, F) = \overset{\circ}{f}(r, w, F)$  for  $(r, w, F) \in \mathbb{R}^2 \times [0, \infty[, \overset{\circ}{f} \geq 0$ , and*

$$w_0 := \sup\{|w| | (r, w, F) \in \text{supp } \overset{\circ}{f}\} < \infty$$

$$F_0 := \sup\{F|(r, w, F) \in \text{supp} \overset{\circ}{f}\} < \infty$$

Let  $\overset{\circ}{\lambda} \in C^1(\mathbb{R})$ ,  $\overset{\circ}{\mu} \in C^2(\mathbb{R})$  with  $\overset{\circ}{\lambda}(r) = \overset{\circ}{\lambda}(r+1)$ ,  $\overset{\circ}{\mu}(r) = \overset{\circ}{\mu}(r+1)$  for  $r \in \mathbb{R}$ , and

$$\overset{\circ}{\mu}'(r) = -4\pi t_0 e^{\overset{\circ}{\lambda} + \overset{\circ}{\mu}} j(r) = -\frac{4\pi^2}{t_0} e^{\overset{\circ}{\lambda} + \overset{\circ}{\mu}} \int_{-\infty}^{\infty} \int_0^{\infty} w \overset{\circ}{f}(r, w, F) dF dw, \quad r \in \mathbb{R}.$$

Then there exists a unique, right maximal, regular solution  $(f, \lambda, \mu)$  of (1.2)-(1.6) with  $(f, \lambda, \mu)(t_0) = (\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})$  on a time interval  $[t_0, T[$  with  $T \in ]t_0, \infty]$ . If

$$\sup \{\mu(t, r) | r \in \mathbb{R}, t \in [t_0, T[\} < \infty$$

then  $T = \infty$ .

This is the content of Thms. 3.3 and 3.4 in [11]. For a *regular solution* all derivatives which appear in the system exist and are continuous by definition, cf. [11].

We now establish a series of estimates which will result in an upper bound on  $\mu$  and will therefore prove that  $T = \infty$ . Similar estimates were used in [1] for the Einstein-Vlasov system with Gowdy symmetry and were generalized to the case of  $T^2$  symmetry in [3]. Unless otherwise specified in what follows constants denoted by  $C$  will be positive, may depend on the initial data and on  $\Lambda$  and may change their value from line to line.

Firstly, integration of (1.4) with respect to  $t$  and the fact that  $p$  is non-negative imply that

$$\begin{aligned} e^{2\mu(t, r)} &= \left[ \frac{t_0(e^{-2\overset{\circ}{\mu}(r)} + k)}{t} - k - \frac{8\pi}{t} \int_{t_0}^t s^2 p(s, r) ds + \frac{\Lambda}{3t}(t^3 - t_0^3) \right]^{-1} \\ &\geq \frac{t}{C - kt + \frac{\Lambda}{3}t^3}, \quad t \in [t_0, T[. \end{aligned} \tag{2.1}$$

In this inequality,  $C$  does not depend on  $\Lambda$ . Next we claim that

$$\int_0^1 e^{\mu+\lambda} \rho(t, r) dr \leq Ct^{-3}, \quad t \in [t_0, T[. \tag{2.2}$$

A lengthy computation shows that

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, r) dr = -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[ 2\rho + q - \frac{\rho+p}{2}(1 + ke^{2\mu} - \Lambda t^2 e^{2\mu}) \right] dr. \tag{2.3}$$

Now using (2.1) we have

$$1 + ke^{2\mu} - \Lambda t^2 e^{2\mu} \leq 1 + \frac{kt - \Lambda t^3}{C - kt + \frac{\Lambda}{3}t^3} = \frac{C - \frac{2}{3}\Lambda t^3}{C - kt + \frac{\Lambda}{3}t^3}.$$

The right hand side of this inequality is negative if  $t \geq (\frac{3C}{2\Lambda})^{1/3}$ . In this case  $1 + ke^{2\mu} - \Lambda t^2 e^{2\mu} \leq 0$  so that, using the fact that  $q \geq 0$  and  $p \geq 0$ , (2.3) implies that

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, r) dr \leq -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[ 2\rho - \frac{\rho}{2}(1 + ke^{2\mu} - \Lambda t^2 e^{2\mu}) \right] dr. \quad (2.4)$$

Setting  $C'(\Lambda) := \frac{3}{\Lambda}(3C - 2k)$ , we have the estimate

$$1 + ke^{2\mu} - \Lambda t^2 e^{2\mu} \leq 1 + \frac{kt - \Lambda t^3}{C - kt + \frac{\Lambda}{3}t^3} \leq C'(\Lambda)t^{-2} - 2,$$

and combining this with (2.4) yields

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, r) dr \leq -\frac{3}{t} \int_0^1 e^{\mu+\lambda} \rho dr + \frac{C'(\Lambda)}{2t^3} \int_0^1 e^{\mu+\lambda} \rho(t, r) dr$$

which multiplied by  $t^3$  gives

$$\frac{d}{dt} \left[ t^3 \int_0^1 e^{\mu+\lambda} \rho dr \right] \leq Ct^{-3} \left[ t^3 \int_0^1 e^{\mu+\lambda} \rho dr \right]$$

By Gronwall's inequality, this implies that

$$\int_0^1 e^{\mu+\lambda} \rho(t, r) dr \leq Ct^{-3},$$

i.e. (2.2) for  $t \geq (\frac{3C}{2\Lambda})^{1/3}$ . For  $t < (\frac{3C}{2\Lambda})^{1/3}$ , (2.3) implies the following, since  $q \geq 0$  :

$$\begin{aligned} \frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, r) dr &\leq -\frac{2}{t} \int_0^1 e^{\mu+\lambda} \rho dr + \frac{1}{t} \int_0^1 e^{\mu+\lambda} \frac{\rho + p}{2} dr \\ &\quad + \frac{1}{t} \int_0^1 \frac{(k - \Lambda t^2)e^{2\mu} e^{\mu+\lambda}}{2} (\rho + p) dr \\ &\leq -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \rho dr, \end{aligned}$$

we have used the fact that  $\rho \geq p$  and  $k - \Lambda t^2 \leq 0$ . By Gronwall's inequality, we obtain

$$\begin{aligned} \int_0^1 e^{\mu+\lambda} \rho(t, r) dr &\leq Ct^{-1} \\ &\leq (Ct^2 + C)t^{-3} \\ &\leq \left[ C(\frac{3C}{2\Lambda})^{2/3} + C \right] t^{-3} \text{ since } t < (\frac{3C}{2\Lambda})^{1/3} \end{aligned}$$

that is (2.2) holds for  $t < (\frac{3C}{2\Lambda})^{1/3}$  as well. Using (2.2) and the equation  $\mu' = -4\pi t e^{\mu+\lambda} j$  we find

$$\begin{aligned} |\mu(t, r) - \int_0^1 \mu(t, \sigma) d\sigma| &= \left| \int_0^1 \int_{\sigma}^r \mu'(t, \tau) d\tau d\sigma \right| \leq \int_0^1 \int_0^1 |\mu'(t, \tau)| d\tau d\sigma \\ &\leq 4\pi t \int_0^1 e^{\mu+\lambda} |j(t, \tau)| d\tau \leq 4\pi t \int_0^1 e^{\mu+\lambda} \rho(t, \tau) d\tau \\ &\leq Ct^{-2}, \quad t \in [t_0, T[, \quad r \in [0, 1]. \end{aligned} \quad (2.5)$$

Next we show that

$$e^{\mu(t, r) - \lambda(t, r)} \leq Ct^{-2}, \quad t \in [t_0, T[, \quad r \in [0, 1] \quad (2.6)$$

To see this observe that by (1.3), (1.4) and (2.1)

$$\begin{aligned} \frac{\partial}{\partial t} e^{\mu-\lambda} &= e^{\mu-\lambda} \left[ 4\pi t e^{2\mu} (p - \rho) + \frac{1 + ke^{2\mu}}{t} - \Lambda t e^{2\mu} \right] \leq e^{\mu-\lambda} \left[ \frac{1 + ke^{2\mu}}{t} - \Lambda t e^{2\mu} \right] \\ &\leq \left[ \frac{1}{t} + \frac{k - \Lambda t^2}{C - kt + \frac{\Lambda}{3}t^3} \right] e^{\mu-\lambda}. \end{aligned}$$

Using the fact that  $-k + \Lambda t^2$  is the derivative of  $C - kt + \frac{\Lambda}{3}t^3$  and integrating this inequality with respect to  $t$  yields

$$e^{\mu-\lambda} \leq C \frac{t}{C - kt + \frac{\Lambda}{3}t^3} \leq Ct^{-2},$$

i.e. (2.6).

We now estimate the average of  $\mu$  over the interval  $[0, 1]$  which in combination with (2.5) will yield the desired upper bound on  $\mu$  :

$$\begin{aligned} \int_0^1 \mu(t, r) dr &= \int_0^1 \dot{\mu}(r) dr + \int_{t_0}^t \int_0^1 \dot{\mu}(s, r) dr ds \\ &\leq C + \int_{t_0}^t \frac{1}{2s} \int_0^1 [e^{2\mu} (8\pi s^2 p + k - \Lambda s^2) + 1] dr ds \\ &\leq C + \frac{1}{2} \ln(t/t_0) + C \int_{t_0}^t s^{-4} ds - \frac{1}{2} \int_{t_0}^t \frac{-k + \Lambda s^2}{C - ks + \frac{\Lambda}{3}s^3} ds \\ &\leq C + \frac{1}{2} \left[ \ln \frac{s}{C - ks + \frac{\Lambda}{3}s^3} \right]_{s=t_0}^{s=t} \end{aligned}$$

where we used (2.1), (2.2), (2.6) and the fact that  $p \leq \rho$ . With (2.5) this implies

$$\mu(t, r) \leq C(1 + t^{-2} + \ln t^{-2}) \leq C, \quad t \in [t_0, T[, \quad r \in [0, 1] \quad (2.7)$$

which by Proposition 2.1 implies  $T = \infty$ . Thus we have proven :

**Theorem 2.2** *For initial data as in Proposition 2.1 the solution of the Einstein-Vlasov system with positive cosmological constant and plane or hyperbolic symmetry, written in areal coordinates, exists for all  $t \in [t_0, \infty[$  where  $t$  denotes the area radius of the surfaces of symmetry of the induced spacetime. The solution satisfies the estimates (2.2), (2.6) and (2.7).*

### 3 On future asymptotic behaviour

In the first part of this section we prove that the spacetime obtained in Theorem 2.2 is timelike and null geodesically complete in the expanding direction. The analogue of this result was proved by Rein, cf. [7], in the case  $\Lambda = 0$ ,  $k = -1$  but with initial data satisfying a certain size restriction, an additional assumption which we drop here due to the fact that  $\Lambda$  does not vanish. The proofs of the results obtained in the first two subsections are modelled on the approach of [7]. In this section we are interested in proving statements about the asymptotic behaviour of solutions at late times. Therefore there is no loss of generality in prescribing data at some large time  $t = t_0 > 0$ .

Firstly we establish a bound on  $w$  along characteristics of the Vlasov equation.

#### 3.1 An estimate along characteristics

Let

$$w_0 := \sup\{|w| | (r, w, F) \in \text{supp} \overset{\circ}{f}\} < \infty,$$

$$F_0 := \sup\{|F| | (r, w, F) \in \text{supp} \overset{\circ}{f}\} < \infty.$$

Except in the vacuum case we have  $w_0 > 0$  and  $F_0 > 0$ . For  $t \geq t_0$  define

$$P_+(t) := \max\{0, \max\{w | (r, w, F) \in \text{supp} f(t)\}\},$$

$$P_-(t) := \min\{0, \min\{w | (r, w, F) \in \text{supp} f(t)\}\}.$$

Fix  $\varepsilon \in ]0, 1[$ . We claim that

$$P_+(t) \leq w_0 \left(\frac{t}{t_0}\right)^{-1+\varepsilon}, \quad P_-(t) \geq -w_0 \left(\frac{t}{t_0}\right)^{-1+\varepsilon}, \quad t \geq t_0. \quad (3.1)$$

Assume that the estimate on  $P_+$  were false for some  $t$ . Define

$$t_1 := \sup \left\{ t \geq t_0 | P_+(s) \leq w_0 \left(\frac{s}{t_0}\right)^{-1+\varepsilon}, t_0 \leq s \leq t \right\}$$

so that  $t_0 \leq t_1 < \infty$  and  $P_+(t_1) = w_0 \left( \frac{t_1}{t_0} \right)^{-1+\varepsilon} > 0$ . Choose  $\alpha \in ]0, 1[$ . By continuity, there exists some  $t_2 > t_1$  such that the following holds :

$$(1 - \alpha)P_+(s) > 0, \quad s \in [t_1, t_2].$$

If for some characteristic curve  $(r(s), w(s), F)$  in the support of  $f$ , that is with  $(r(t_0), w(t_0), F) \in \text{supp } \overset{\circ}{f}$ , and for some  $t \in ]t_1, t_2]$  the estimate

$$(1 - \alpha/2)P_+(t) \leq w(t) \leq P_+(t) \quad (3.2)$$

holds then

$$(1 - \alpha)P_+(s) \leq w(s) \leq P_+(s), \quad s \in [t_1, t]. \quad (3.3)$$

Note that the estimates on  $w$  from above hold by definition of  $P_+$  in any case. Let  $(r(s), w(s), F)$  be a characteristic in the support of  $f$  satisfying (3.2) for some  $t \in ]t_1, t_2]$  and thus (3.3) on  $[t_1, t]$ . Then on  $[t_1, t]$ ,

$$\begin{aligned} \dot{w} &= \frac{4\pi^2}{s} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left( \tilde{w} \sqrt{1 + w^2 + F/s^2} - w \sqrt{1 + \tilde{w}^2 + \tilde{F}/s^2} \right) f d\tilde{F} d\tilde{w} \\ &\quad + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} sw e^{2\mu} \\ &\leq \frac{4\pi^2}{s} e^{2\mu} \int_0^{P_+(s)} \int_0^{F_0} \frac{\tilde{w}^2(1 + w^2 + F/s^2) - w^2(1 + \tilde{w}^2 + \tilde{F}/s^2)}{\tilde{w} \sqrt{1 + w^2 + F/s^2} + w \sqrt{1 + \tilde{w}^2 + \tilde{F}/s^2}} f d\tilde{F} d\tilde{w} \\ &\quad + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} sw e^{2\mu} \\ &\leq \frac{4\pi^2}{s} e^{2\mu} \int_0^{P_+(s)} \int_0^{F_0} \frac{\tilde{w}(1 + F)}{w} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w - \frac{\Lambda}{2} sw e^{2\mu} \\ &\leq 4\pi^2 F_0 (1 + F_0) \| \overset{\circ}{f} \| \frac{e^{2\mu}}{2s} P_+^2(s) \frac{1}{w} + \frac{1 + ke^{2\mu} - \Lambda s^2 e^{2\mu}}{2s} w \quad (3.4) \\ &\leq 4\pi^2 F_0 (1 + F_0) \| \overset{\circ}{f} \| \frac{1}{(1 - \alpha)^2} \frac{e^{2\mu}}{2s} w + \frac{1 + ke^{2\mu} - \Lambda s^2 e^{2\mu}}{2s} w, \text{ using (3.3)} \\ &\leq \frac{1 + (C + k - \Lambda s^2) e^{2\mu}}{2s} w. \end{aligned}$$

Since  $s$  is large,  $C + k - \Lambda s^2$  is negative so that using (2.1) we have

$$\dot{w} \leq \frac{1 + \frac{Cs + ks - \Lambda s^3}{C - ks + \frac{\Lambda}{3}s^3}}{2s} w. \quad (3.5)$$

Now

$$\begin{aligned} 3 + \frac{Cs + ks - \Lambda s^3}{C - ks + \frac{\Lambda}{3}s^3} &= \frac{3C + Cs - 2ks}{C - ks + \frac{\Lambda}{3}s^3} \\ &\leq \left( \frac{9C}{\Lambda} s^{-1} + \frac{3C}{\Lambda} - \frac{6k}{\Lambda} \right) s^{-2} \\ &\leq \frac{C - 6k}{\Lambda} s^{-2} \end{aligned}$$

so that setting  $C'(\Lambda) := \frac{C-6k}{\Lambda}$  we obtain the estimate

$$1 + \frac{Cs + ks - \Lambda s^3}{C - ks + \frac{\Lambda}{3}s^3} \leq C'(\Lambda)s^{-2} - 2. \quad (3.6)$$

Thus (3.5) implies that

$$\dot{w} \leq -\frac{w}{s} + \frac{C'(\Lambda)}{2}s^{-3}w$$

which multiplied by  $s^{1-\varepsilon}$  gives

$$\begin{aligned} \frac{d}{ds}(s^{1-\varepsilon}w) &\leq s^{1-\varepsilon}w \left( -\varepsilon s^{-1} + \frac{C'(\Lambda)}{2}s^{-3} \right) \\ &\leq 0 \text{ since } s \text{ is large.} \end{aligned}$$

Thus the function  $s \mapsto s^{1-\varepsilon}w(s)$  is decreasing on  $[t_1, t]$ . This implies that

$$t^{1-\varepsilon}w(t) \leq t_1^{1-\varepsilon}w(t_1) \leq t_1^{1-\varepsilon}P_+(t_1) = \frac{w_0}{t_0^{-1+\varepsilon}}$$

by assumption on  $t_1$  and so

$$w(t) \leq w_0 \left( \frac{t}{t_0} \right)^{-1+\varepsilon}. \quad (3.7)$$

This estimate holds only for characteristics which satisfy (3.2), but this is sufficient to conclude that

$$P_+(t) \leq w_0 \left( \frac{t}{t_0} \right)^{-1+\varepsilon}, \quad t \in [t_1, t_2],$$

in contradiction to the choice of  $t_1$ . The estimate on  $P_+$  is now established. The analogous arguments for characteristics with  $w < 0$  yield the assertion for  $P_-$ .

Next we consider characteristics which are not in the support of  $f$ . We can rewrite the inequality (3.4) for  $s \in [t_0, t]$  and  $w(s) > 0$  :

$$\dot{w} \leq 4\pi^2 F_0(1 + F_0) \| \overset{\circ}{f} \| \frac{e^{2\mu}}{2s} P_+^2(s) \frac{1}{w} + \frac{1 + (k - \Lambda s^2)e^{2\mu}}{2s} w.$$

From (2.1) and (3.6) it follows that  $\frac{1 + (k - \Lambda s^2)e^{2\mu}}{2s} \leq 0$ . Using the estimate (3.1) on  $P_+$  and (2.7) we obtain

$$\dot{w} \leq C s^{2\varepsilon-3} \frac{1}{2w}.$$

Hence

$$\frac{d}{ds}(w^2) \leq C s^{2\varepsilon-3}.$$

Integrating this over  $[t_0, t]$  yields

$$w^2(t) \leq C, \quad t \geq t_0. \quad (3.8)$$

The analogous arguments for characteristics outside the support of  $f$  with  $w < 0$  yield the same estimate. Thus by (3.7), (3.1) and (3.8) we can state :

**Proposition 3.1** *For any characteristic  $(r, w, F)$ , for any solution of Einstein-Vlasov system with positive cosmological constant and plane or hyperbolic symmetry written in areal coordinates and with initial data as in Proposition 2.1,*

$$|w(t)| \leq C, \quad t \geq t_0,$$

where the positive constant  $C$  depends on the initial data.

### 3.2 Geodesic completeness

Let  $\tau_- \in \exists \tau \mapsto (x^\alpha(\tau), p^\beta(\tau))$  be a geodesic whose existence interval is maximally extended and such that  $x^0(\tau_0) = t(\tau_0) = t_0$  for some  $\tau_0 \in ]\tau_-, \tau_+[$ . We want to show that for future-directed timelike and null geodesics  $\tau_+ = +\infty$ . Consider first the case of a timelike geodesic, i.e.,

$$g_{\alpha\beta} p^\alpha p^\beta = -m^2 ; \quad p^0 > 0$$

with  $m > 0$ . Since  $\frac{dt}{d\tau} = p^0 > 0$ , the geodesic can be parametrized by the coordinate time  $t$ . With respect to coordinate time the geodesic exists on the interval  $[t_0, \infty[$  since on bounded  $t$ -intervals the Christoffel symbols are bounded and the right hand sides of the geodesic equations written in coordinate time are linearly bounded in  $p^1, p^2, p^3$ . Recall that along geodesics the variables  $t, r, p^0, w := e^\lambda p^1, F := t^4 [(p^2)^2 + \sin_k^2 \theta (p^3)^2]$  satisfy the following system of differential equations :

$$\frac{dr}{d\tau} = e^{-\lambda} w, \quad \frac{dw}{d\tau} = -\dot{\lambda} p^0 w - e^{2\mu-\lambda} \mu' (p^0)^2, \quad \frac{dF}{d\tau} = 0 \quad (3.9)$$

$$\frac{dt}{d\tau} = p^0, \quad \frac{dp^0}{d\tau} = -\dot{\mu} (p^0)^2 - 2e^{-\lambda} \mu' p^0 w - e^{-2\mu} \dot{\lambda} w^2 - e^{-2\mu} t^{-3} F. \quad (3.10)$$

Along the geodesic we define  $w$  and  $F$  as above. Then the relation between coordinate time and proper time along the geodesic is given by

$$\frac{dt}{d\tau} = p^0 = e^{-\mu} \sqrt{m^2 + w^2 + F/t^2},$$

and to control this we need to control  $w$  as a function of coordinate time. By (2.1) we have the estimate

$$e^\mu \geq Ct^{-1}, \quad t \geq t_0.$$

Combining this with the estimate on  $w$  in Proposition 3.1 yields the following along the geodesic :

$$\frac{d\tau}{dt} = \frac{e^\mu}{\sqrt{m^2 + w^2 + F/t^2}} \geq \frac{Ct^{-1}}{\sqrt{m^2 + C + F}}.$$

Since the integral of the right hand side over  $[t_0, \infty[$  diverges,  $\tau_+ = +\infty$  as desired. In the case of a future-directed null geodesic, i.e.  $m = 0$  and  $p^0(\tau_0) > 0$ ,  $p^0$  is everywhere positive and the quantity  $F$  is again conserved. The argument can now be carried out exactly as in the timelike case, implying that  $\tau_+ = +\infty$ . We have proven :

**Theorem 3.2** *Consider initial data with plane or hyperbolic symmetry for the Einstein-Vlasov system with positive cosmological constant. Suppose that the regularity properties required in the statement of Proposition 2.1 are satisfied. If the gradient of  $R$  is initially past-pointing then there is a corresponding Cauchy development which is future geodesically complete.*

### 3.3 Determination of the leading asymptotic behaviour

In this subsection we determine the explicit leading behaviour of  $\lambda$ ,  $\mu$ ,  $\dot{\lambda}$ ,  $\dot{\mu}$ ,  $\mu'$ , and later on we compute the generalized Kasner exponents and prove that each of them tends to  $1/3$  as  $t$  tends to  $+\infty$ .

Let us recall briefly some of the relevant notation. Let  $I$  be a set of real numbers and  $t_*$  a real number or infinity. The asymptotic behaviour of a function  $g$  defined on  $I$  as  $t \rightarrow t_*$  is to be described. It will be compared with a positive function  $h(t)$ , typically a power of  $t$ . The notation  $g(t) = O(h(t))$  as  $t \rightarrow t_*$  means that there is a neighbourhood  $U$  of  $t_*$  such that there is a constant  $C$  with  $|g(t)| \leq Ch(t)$  for all  $t$  belonging to both  $I$  and  $U$ . The notation  $g(t) = o(h(t))$  as  $t \rightarrow t_*$  means that  $g(t)/h(t)$  tends to 0 as  $t \rightarrow t_*$ .

Now (1.4) can be written in the form

$$\frac{d}{dt}(te^{-2\mu}) = \Lambda t^2 - k - 8\pi t^2 p. \quad (3.11)$$

Integrating this over  $[t_0, t]$  yields

$$te^{-2\mu} = (t_0 e^{-2\mu(t_0)} + kt_0 - \frac{\Lambda}{3}t_0^3) + \frac{\Lambda}{3}t^3 - kt - \int_{t_0}^t 8\pi s^2 p ds. \quad (3.12)$$

By (3.7) we have the following, where  $C$  is a positive constant and  $\varepsilon \in ]0, 1[$  :

$$w \leq Ct^{-1+\varepsilon} \text{ for } t \geq t_0.$$

Using the expression (1.8) for  $p$ , this implies that

$$p \leq Ct^{-5+3\varepsilon}$$

so that

$$8\pi t^2 p \leq Ct^{-3+3\varepsilon}. \quad (3.13)$$

Assuming  $\varepsilon < 2/3$  we obtain, using (3.12)

$$|te^{-2\mu} - \frac{\Lambda}{3}t^3 + kt| \leq C,$$

i.e.,

$$e^{-2\mu} = \frac{\Lambda}{3}t^2 (1 + O(t^{-2})).$$

It follows that

$$e^\mu = \sqrt{\frac{3}{\Lambda}}t^{-1} (1 + O(t^{-2})). \quad (3.14)$$

Now by (1.3), and using (3.14) and the fact that  $8\pi t\rho = O(t^{-2+\varepsilon})$ , we have

$$\begin{aligned} \dot{\lambda} &= \frac{1}{2}(\Lambda t + 8\pi t\rho)e^{2\mu} - \frac{1+ke^{2\mu}}{2t} \\ &= \frac{3}{2\Lambda}t^{-2}(1+O(t^{-2}))(\Lambda t + O(t^{-2+\varepsilon})) - \frac{1}{2t} - \frac{k}{2t}\left[\frac{3}{\Lambda}t^{-2}(1+O(t^{-2}))\right] \end{aligned}$$

and hence

$$\dot{\lambda} = t^{-1}(1+O(t^{-2})). \quad (3.15)$$

Integrating this over  $[t_0, t]$  yields

$$\lambda = \ln t [1 + O((\ln t)^{-1})]. \quad (3.16)$$

Next, using (1.4), (3.13) and (3.14) we have

$$\dot{\mu} = -t^{-1}(1+O(t^{-2})) \quad (3.17)$$

and integrating this over  $[t_0, t]$  yields

$$\mu = -\ln t [1 + O((\ln t)^{-1})]. \quad (3.18)$$

Now (3.16) implies that

$$e^\lambda = O(t),$$

the expression (1.9) of  $j$  implies that

$$|j| \leq Ct^{-4+2\varepsilon}$$

and thus using equation (1.5) we obtain

$$\mu' = O(t^{-3+2\varepsilon}). \quad (3.19)$$

We can now compute the limiting values of the generalized Kasner exponents namely

$$\frac{K_1^1(t, r)}{K(t, r)} = \frac{t\dot{\lambda}(t, r)}{t\dot{\lambda}(t, r) + 2}, \quad \frac{K_2^2(t, r)}{K(t, r)} = \frac{K_3^3(t, r)}{K(t, r)} = \frac{1}{t\dot{\lambda}(t, r) + 2},$$

where  $K(t, r) = K_i^i(t, r)$  is the trace of the second fundamental form  $K_{ij}$  of the metric. We refer to [6] for the computation. Using (3.15), we see that as  $t$  tends to  $+\infty$ , each of those quantities tends to  $1/3$  uniformly in  $r \in \mathbb{R}$ . We have proved the following :

**Theorem 3.3** *Let  $(f, \lambda, \mu)$  be a solution of the Einstein-Vlasov system with plane or hyperbolic symmetry and  $\Lambda > 0$  given in the expanding direction. Then the following properties hold at late times : (3.15), (3.16), (3.17), (3.18), (3.19), with  $\varepsilon \in ]0, 2/3[$  ; and*

$$\lim_{t \rightarrow +\infty} \frac{K_1^1(t, r)}{K(t, r)} = \lim_{t \rightarrow +\infty} \frac{K_2^2(t, r)}{K(t, r)} = \lim_{t \rightarrow +\infty} \frac{K_3^3(t, r)}{K(t, r)} = \frac{1}{3}.$$

This theorem shows how the de Sitter solution acts as a model for the dynamics of the class of solutions considered in this paper. For if we set  $\lambda = \ln t$ ,  $\mu = -\ln t$  and  $k = 0$  the spacetime obtained is the de Sitter solution. Thus the leading terms in the asymptotic expansions of the metric components are exactly the quantities defined by the de Sitter spacetime.

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